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Nonlinear Bifurcation*

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1. INTRODUCTION

We consider problems of the form

$$\begin{aligned} &\text{(a)} \qquad Lu + \lambda g(\lambda, x, u) = 0, \qquad x \in D; \\ &\text{(b)} \qquad Bu = 0, \qquad x \in \partial D, \end{aligned} \tag{1.1}$$

for a very wide class of linear elliptic operators L, say of order $2m$, and linear boundary operators B, of order m. For convenience only, we take the linear problem to be self-adjoint. The nonlinearity, $g(\lambda, x, z)$, is assumed to satisfy, for all λ in some interval, $\mathscr{I}$, and all $x \in D$:

$$\text{(a) } g(\lambda, x, 0) \equiv 0, \qquad \text{(b) } g_\lambda(\lambda, x, 0) \equiv 0, \qquad \text{(c) } g_z(\lambda, x, 0) > 0; \tag{1.2}$$

and appropriate smoothness conditions. It is clear that the trivial solution, $u(x) \equiv 0$, is a solution of (1.1) for all $\lambda \in \mathscr{I}$. In the case where $g(\lambda, x, z) \equiv f(x, z)$ is independent of λ it is well-known that nontrivial solutions can only bifurcate from the eigenvalues, μ_j, of the corresponding linearized problem:

$$L\psi + \mu f_z(x, 0)\, \psi = 0 \text{ in } D; \;\; B\psi = 0 \text{ on } \partial D. \tag{1.3}$$

The fact that bifurcations actually do occur at some of these eigenvalues will follow from the more general problem treated here. Such results have been obtained for similar problems by Krasnoselsky [4], Vainberg [6], M. Berger [2] and many others.

In the case of nonlinear dependence on λ we consider for each fixed $\lambda \in \mathscr{I}$ the eigenvalue problem:

$$L\phi + \mu g_z(\lambda, x, 0)\, \phi = 0 \text{ on } D; \;\; B\phi = 0 \text{ on } \partial D. \tag{1.4}$$

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The eigenvalues $\mu_j \equiv \mu_j(\lambda)$, $j = 1, 2,...$ are defined for all $\lambda \in \mathscr{I}$ and for each j we denote the (real) roots of

$$\lambda = \mu_j(\lambda) \tag{1.5}$$

by λ_{jk}, $k = 1, 2,..., k_j$ for $j = 1, 2,....$ The corresponding eigenfunctions are, say, $\phi(\lambda_{jk}, x)$. We shall show that *a bifurcation of* (1.1) *occurs at each value of* $\lambda = \lambda_{jk} \in \mathscr{I}$ *for which* $\mu_j(\lambda_{jk})$ *is a simple eigenvalue of* (1.4). Bifurcation also occurs from each λ_{jk} for which $\mu_j(\lambda_{jk})$ is an eigenvalue of *odd* multiplicity. But as our proof of this fact is not constructive, we do not include it here.

For a more detailed description of our results let $\lambda_0 = \lambda_{jk}$ denote any such root of (1.5) and set $\phi_0(x) \equiv \phi(\lambda_0, x)$. Then it clearly follows that

$$L\phi_0 + \lambda_0 g_z(\lambda_0, x, 0)\,\phi_0 = 0 \text{ on } D, \; B\phi_0 = 0 \text{ on } \partial D. \tag{1.6}$$

The eigenfunction $\phi_0(x)$ can be made unique, to within its sign, by imposing the normalization condition

$$\int_D g_z(\lambda_0, x, 0)\,\phi_0^2(x)\,dx = 1. \tag{1.7}$$

For specific positive constants ϵ_0, m_0 and M_0 we will show that for each ϵ in $0 < |\epsilon| \leqslant \epsilon_0$ the problem (1.1) has a nontrivial solution of the form

$$\begin{aligned} &\text{(a)} \qquad u \equiv u(\epsilon, x) = \epsilon\phi_0(x) + \epsilon^2\phi_1(\epsilon, x), \qquad |\phi_1(\epsilon, x)| \leqslant M_0 \\ &\text{(b)} \qquad \lambda \equiv \lambda(\epsilon) = \lambda_0 + \epsilon\lambda_1(\epsilon), \qquad |\lambda_1(\epsilon)| \leqslant m_0, \end{aligned} \tag{1.8}$$

where ϕ_1 is orthogonal to ϕ_0 in the sense

$$\text{(c)} \qquad \int_D g_z(\lambda_0, x, 0)\,\phi_0(x)\,\phi_1(\epsilon, x)\,dx = 0. \tag{1.8}$$

These solutions are the unique nontrivial solutions of sufficiently small magnitude for λ near λ_0. It is also shown that the transformation

$$[\epsilon, \phi_0(x)] \to [-\epsilon, -\phi_0(x)]$$

yields the same family of solutions and thus the sign ambiguity in the choice of $\phi_0(x)$ is removed. The constants ϵ_0, m_0, and M_0 are easily determined for many problems [see (2.10)].

The above results require that $g_{zz}(\lambda, x, z)$ and $g_{\lambda z}(\lambda, x, z)$ be continuous on an appropriate bounded domain and that $|g_{\lambda z}(\lambda, x, z)|$ be sufficiently small on this domain. If these derivatives are Lipschitz continuous then we easily show that for some positive constants m_2 and $\epsilon_0' \leqslant \epsilon_0$:

$$\lambda(\epsilon) = \lambda_0 + \epsilon\lambda_1(0) + \epsilon^2\lambda_2(\epsilon), \qquad |\lambda_2(\epsilon)| \leqslant m_2, \tag{1.9a}$$

for all ϵ in $|\epsilon| \leqslant \epsilon_0'$ where

$$\lambda_1(0) = -\lambda_0 \frac{\frac{1}{2}\int_D g_{zz}(\lambda_0, x, 0)\, \phi_0^3(x)\, dx}{1 + \lambda_0 \int_D g_{\lambda z}(\lambda_0, x, 0)\, \phi_0^2(x)\, dx}. \tag{1.9b}$$

If $\lambda_1(0) \neq 0$ we see by taking ϵ in $0 < |\epsilon| \leqslant |\lambda_1(0)|/m_2$ that (1.1) has solutions bifurcating from λ_0 for all λ in some λ interval with λ_0 as an interior point.

If $\lambda_1(0) = 0$ it is possible that bifurcated solutions (of small norm) exist only for $\lambda > \lambda_0$ or else for $\lambda < \lambda_0$. This occurs, for instance, if $g_{zz}(\lambda_0, x, 0) \equiv 0$ and $g_{zzz}(\lambda_0, x, 0)$ is of one sign on D. More generally, we show that if $\partial^k g(\lambda_0, x, 0)/\partial z^k \equiv 0$ for $k = 2, 3,..., p-1$ and the p-th derivative satisfies a weak Lipschitz condition then for some constants m_p and ϵ_0':

$$\lambda(\epsilon) = \lambda_0 + \epsilon^{p-1}\lambda_{p-1}(0) + \epsilon^p \lambda_p(\epsilon), \qquad |\lambda_p(\epsilon)| \leqslant m_p, \tag{1.10a}$$

for all ϵ in $|\epsilon| \leqslant \epsilon_0'$, where

$$\lambda_{p-1}(0) = \frac{-\lambda_0}{p!} \frac{\int_D [\partial^p g(\lambda_0, x, 0)/\partial z^p]\, \phi_0^{p+1}(x)\, dx}{1 + \lambda_0 \int_D g_{\lambda z}(\lambda_0, x, 0)\, \phi_0^2(x)\, dx}. \tag{1.10b}$$

The case of bifurcated solutions existing only on one side of λ_0 holds if p is odd and $\lambda_{p-1}(0) \neq 0$. In fact then two distinct nontrivial solutions, with small norm, exist for each λ in some interval above or below λ_0 (with λ_0 as an endpoint). For p even and $\lambda_{p-1}(0) \neq 0$ the point λ_0 is interior to some interval of λ values containing bifurcated solutions of (1.1).

Our results are similar to some of those due to Krasnoselski [4] p. 191–228 for integral equations, to J. Keller [3] p. 26–40 for ordinary differential equations and to Berger for elliptic equations [2] p. 127–133. We do not employ topological methods or implicit function theorems but simply a contracting mapping (somewhat differently from that used in the Schmidt bifurcation theory, see [2].) That is we show that $\phi_1(\epsilon, x)$ and $\lambda_1(\epsilon)$ are determined by a specific iteration scheme which converges for each $\epsilon \neq 0$ in $|\epsilon| < \epsilon_0$. Then we analyze $\lambda_1(\epsilon)$ in detail. The form of solution presented in (1.8) is essentially suggested by standard perturbation theory. Thus our results justify such procedures up to second order in many cases and we obtain explicit lower bounds on their range of validity and upper bounds on their deviation from the exact solution. It is clear, although quite complicated in the details, that our analysis could also be employed to justify higher order perturbation expansions. Extensions to systems and non-simple eigenvalues are suggested and will be reported elsewhere.

2. An Iteration Scheme

Let λ_0 and $\phi_0(x)$ be a simple "eigenvalue" and corresponding normalized eigenfunction satisfying (1.5)–(1.7). Then for any ϵ we introduce the notation

$$V(\epsilon, x) \equiv \epsilon\phi_0(x) + \epsilon^2 v(x), \; \Lambda(\epsilon) \equiv \lambda_0 + \epsilon\eta(\epsilon), \tag{2.1}$$

where conditions on $v(x)$ and $\eta(\epsilon)$ will be imposed later. Using (1.6) it follows that $u = V$ and $\lambda = \Lambda$ will be a solution of (1.1) if, and only if, v and η satisfy:

$$\begin{aligned} &Lv + \lambda_0 g_z(\lambda_0, x, 0)v \\ &\quad = \epsilon^{-2}\{\lambda_0[g_z(\lambda_0, x, 0)\, V(\epsilon, x) - g(\Lambda, x, V)] - \epsilon\eta(\epsilon)\, g(\Lambda, x, V)\} \qquad \text{on} \quad D \\ &Bv = 0 \qquad \text{on} \quad \partial D. \end{aligned} \tag{2.2}$$

However since λ_0 is a simple eigenvalue of (1.6) the above boundary value problem can have a solution if, and only if, the right side satisfies the orthogonality condition:

$$\int_D \phi_0(x)(\lambda_0\{g_z(\lambda_0, x, 0)\, V(\epsilon, x) - g[\Lambda, x, V(\epsilon, X)]\} - \epsilon\eta(\epsilon)\, g[\Lambda, x, V(\epsilon, x)])\, dx = 0. \tag{2.3}$$

It is only here that we require the self-adjointness of (L, B). Of course we could easily drop this requirement and employ the eigenfunction of the adjoint problem; but we do not bother.

Under appropriate conditions we proceed to show that a solution $v(x)$, $\eta(\epsilon)$ of (2.2) satisfying (2.3) exists for sufficiently small $|\epsilon|$. More precisely this solution is the limit of the iterates $\{v^{(\nu)}\}$, $\{\eta^{(\nu)}\}$ defined by: $v^{(0)} \equiv 0$, $\eta^{(0)} = 0$ and, for $\nu = 0, 1, 2,\ldots$

$$\begin{aligned} &Lv^{(\nu+1)} + \lambda_0 g_z(\lambda_0, x, 0)\, v^{(\nu+1)} \\ &\quad = \epsilon^{-2}\{\lambda_0[g_z(\lambda_0, x, 0)\, V^{(\nu)} - g(\Lambda^{(\nu)}, x, V^{(\nu)})] - \epsilon\eta^{(\nu+1)} g(\Lambda^{(\nu)}, x, V^{(\nu)})\}, \\ &Bv^{(\nu+1)} = 0, \end{aligned} \tag{2.4}$$

where

$$\epsilon\eta^{(\nu+1)} = \lambda_0 \frac{\int_D \phi_0(x)[g_z(\lambda_0, x, 0)\, V^{(\nu)} - g(\Lambda^{(\nu)}, x, V^{(\nu)})]\, dx}{\int_D \phi_0(x)\, g(\Lambda^{(\nu)}, x, V^{(\nu)})\, dx}. \tag{2.5}$$

We have employed the obvious notation

$$V^{(\nu)}(\epsilon, x) \equiv \epsilon\phi_0(x) + \epsilon^2 v^{(\nu)}(x), \quad \Lambda^{(\nu)} \equiv \lambda_0 + \epsilon\eta^{(\nu)}(\epsilon).$$

Since the boundary value problem (2.4) for the determination of $v^{(\nu+1)}$ is linear and, by virtue of (2.5), the right hand side is orthogonal to $\phi_0(x)$, we can make the iterate $v^{(\nu+1)}$ unique by imposing the orthogonality condition

$$\int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, v^{(\nu+1)}(\epsilon, x)\, dx = 0. \tag{2.6}$$

Of course we use here the assumption that λ_0 is a simple eigenvalue.

It should be observed that the iteration scheme (2.4) is simply

$$LV^{(\nu+1)} + \lambda_0 g_z(\lambda_0, x, 0) V^{(\nu+1)} = \lambda_0 g_z(\lambda_0, x, 0) V^{(\nu)} - \Lambda^{(\nu+1)} g(\Lambda^{(\nu)}, x, V^{(\nu)}).$$

This is the "chord method" of [7] or the "special Newton method" of [8].

To clarify the arguments and simplify the notation we shall study the convergence of the sequences $\{v^{(\nu)}\}$ and $\{\eta^{(\nu)}\}$ formally by means of contracting maps. First we list some smoothness and other assumptions regarding the operators L and B, the domain D and the nonlinearity $g(\lambda, x, z)$. As is already implied we assume that the coefficients in the $2m$-th order linear elliptic operator L and the m-th order linear boundary operator B and the boundary ∂D, are so smooth that the problem

$$\begin{aligned} Lv + \lambda_0 g_z(\lambda_0, x, 0) v &= \psi(x) \qquad \text{on} \quad D, \\ Bv &= 0 \qquad \text{on} \quad \partial D, \end{aligned} \tag{2.7a}$$

has a solution $v(x) \in C^{2m+\alpha}(D)$ provided $\psi(x) \in C^{\alpha}(D)$ and

$$\int_D \phi_0(x)\, \psi(x)\, dx = 0; \tag{2.7b}$$

see for example Ladyženskaja and Ural'tseva [5], Theorem 3.2, p. 137. By $C^{r+\alpha}(D)$ we denote the class of functions with continuous derivatives up to order r whose r-th order derivatives satisfy on $\bar{D}$ a uniform Hölder condition with exponent α, $0 < \alpha < 1$.

Further we make the stronger assumption that when the solution $v(x)$ of (2.7) is made unique by the orthogonality condition

$$\int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, v(x)\, dx = 0, \tag{2.7c}$$

there exists some constant $\tilde{G}$ such that

$$\| v \| \leqslant \tilde{G} \| \psi \|, \tag{2.8}$$

in the maximum norm: $\| v \| = \sup_{x \in \bar{D}} | v(x)|$. If, for example, there exists a generalized Green's function, $G(x, y)$, determined by

$$\begin{aligned} LG + \lambda_0 g_z(\lambda_0, x, 0)\, G &= \delta(x - y) - \phi_0(x)\, \phi_0(y), \quad x \text{ and } y \text{ on D;} \\ BG &= 0, \quad x \text{ on } \partial D, \quad y \text{ on } D; \end{aligned}$$

then we can take

$$\tilde{G} = \left\| \int_D |G(x, y)| \, dy \right\|.$$

However estimates of the form (2.8), but in a different norm, can be obtained without reference to generalized Green's functions. We indicate later a simple extension of a result in Agmon, Douglis, and Nirenberg [1] which yields such estimates for very general operators.

To impose conditions on $g(\lambda, x, z)$ and its derivatives we define the $n + 2$ dimensional domain

$$S_0 \equiv \{(\lambda, x, z) \mid |\lambda - \lambda_0| \leq 1, \qquad x \in \bar{D}, \qquad |z| \leq \epsilon_2(1 + \|\phi_0\|)\}, \tag{2.9a}$$

where $\|\phi_0\|$ is as defined above and ϵ_2 is some fixed but *arbitrary* positive number. For functions defined on S_0 we emply the maximum norm, say,

$$\|g\| \equiv \sup_{(\lambda,x,z)\in S_0} |g(\lambda, x, z)|. \tag{2.9b}$$

We shall always assume that the interval $[\lambda_0 - 1, \lambda_0 + 1] \subset \mathscr{I}$. Now we define the constants, with $\tilde{D} \equiv \int_D dx$,

$$\begin{aligned}
&\text{(a)} \qquad m_0 \equiv 2|\lambda_0| \tilde{D} \|\phi_0\| (1 + \|\phi_0\|)^2 \|g_{zz}\|, \\
&\text{(b)} \qquad M_0 \equiv \tilde{G} m_0 \frac{1 + 2\tilde{D} \|\phi_0\| (1 + \|\phi_0\|) \|g_z\|}{2\tilde{D} \|\phi_0\|} \\
&\text{(c)} \qquad \epsilon_1 \equiv \min\{m_0^{-1}, M_0^{-1}, |\lambda_0| m_0^{-1}, \epsilon_2\}, \\
&\text{(d)} \qquad \epsilon_0 \equiv \min\left\{\frac{1}{3m_0}, \frac{1}{4(1 + \|\phi_0\|)M_0}, \epsilon_1\right\}.
\end{aligned} \tag{2.10}$$

The optimal choice for ϵ_2 is that which maximizes ϵ_1. However, since m_0 and M_0 are nondecreasing functions of ϵ_2 (through the norms of g_z and g_{zz} over S_0) it is clear that ϵ_1 is a nonincreasing function of ϵ_2 provided ϵ_2 is sufficiently large and that ϵ_1 is a nondecreasing function of ϵ_2 provided ϵ_2 is sufficiently small. Thus, there is a unique optimal value of ϵ_2 for which

$$\epsilon_1 = \epsilon_2 = \min\{m_0^{-1}, M_0^{-1}, |\lambda_0| m_0^{-1}\}$$

In actual applications a few trial choices for ϵ_2 in (2.9a) yield, after determining the corresponding ϵ_1 of (2.10c), reasonable approximations to the optimal value.

In addition to conditions (1.2) we assume that

$$g(\lambda, x, z) \in C^\alpha(S_0), g_z(\lambda_0, x, 0) \in C^\alpha(D), g_{zz}(\lambda, x, z) \in C(S_0), g_{\lambda z}(\lambda, x, z) \in C(S_0). \tag{2.11}$$

Finally, we require that in the norm (2.9).

$$\| g_{\lambda z} \| \leqslant \frac{\min\{1, m_0/M_0\}}{4|\lambda_0| \tilde{D}(1 + \|\phi_0\|)^2}. \tag{2.12}$$

As will be apparent in the analysis we have not sought the best constants or weakest conditions but rather values that simplify the estimates and still yield reasonably "large" values for ϵ_0.

To formulate the contracting map we introduce the set of functions

$$\mathscr{A}_0 \equiv \left\{ v(x) \mid v(x) \in C^{2m+\alpha}(D), \| v \| \leqslant M_0, \int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, v(x)\, dx = 0 \right\}, \tag{2.13a}$$

and the interval

$$\mathscr{I}_0 \equiv \{\eta \mid |\eta| \leqslant m_0\}. \tag{2.13b}$$

Using the notation (2.1) we let

$$\begin{aligned} &\text{(a)} \quad \Delta(\Lambda, V) \equiv \lambda_0 \frac{\int_D \phi_0(x)[g_z(\lambda_0, x, 0)\, V - g(\Lambda, x, V)]\, dx}{\int_D \phi_0(x)\, g(\Lambda, x, V)\, dx}, \\ &\text{(b)} \quad R(\Lambda, V) \equiv \epsilon^{-2}\{\lambda_0[g_z(\lambda_0, x, 0)\, V - g(\Lambda, x, V)] - \Delta(\Lambda, V)\, g(\Lambda, x, V)\}. \end{aligned} \tag{2.14}$$

For each $v(x) \in \mathscr{A}_0$ and $\eta \in \mathscr{I}_0$ a nonlinear transformation, T_ϵ, is defined for each ϵ in $0 < |\epsilon| \leqslant \epsilon_1$ by

$$T_\epsilon[\eta, v(x)] = [\tilde{\eta}, \tilde{v}(x)], \tag{2.15a}$$

where

$$\tilde{\eta} = \epsilon^{-1}\Delta(\Lambda, V), \tag{2.15b}$$

and $\tilde{v}(x)$ is the unique solution of

$$\begin{aligned} &\text{(c)} \quad L\tilde{v} + \lambda_0 g_z(\lambda_0, x, 0)\, \tilde{v} = R(\Lambda, V) \quad \text{on} \quad D, \qquad B\tilde{v} = 0 \quad \text{on} \quad \partial D; \\ &\text{(d)} \qquad \int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, \tilde{v}(x)\, dx = 0. \end{aligned} \tag{2.15}$$

The iteration scheme formulated in (2.4)–(2.5) is simply

$$T_\epsilon[\eta^{(\nu)}, v^{(\nu)}(x)] = [\eta^{(\nu+1)}, v^{(\nu+1)}(x)]; \ \nu = 0, 1, \cdots. \tag{2.16}$$

We will show that T_ϵ takes $\mathscr{I}_0 \times \mathscr{A}_0$ into itself for each ϵ in $0 < |\epsilon| \leqslant \epsilon_1$ and that it is contracting for $|\epsilon| \leqslant \epsilon_0 \leqslant \epsilon_1$. It is then clear that any initial iterate $[\eta^{(0)}, v^{(0)}(x)] \in \mathscr{I}_0 \times \mathscr{A}_0$ can be employed in (2.4) provided that $|\epsilon| \leqslant \epsilon_0$.

3. Convergence Proof

We assume throughout this section that: $\eta \in \mathscr{I}_0$, $v(x) \in \mathscr{A}_0$ and $|\epsilon| \leqslant \epsilon_1$. Then with the notation (2.1) it follows from (2.10c) and (2.13) that $(\Lambda(\epsilon), x, V(\epsilon, x)) \in S_0$ for all $x \in \bar{D}$. Thus, we may employ the continuity conditions in (2.11), (1.2a) and the identity

$$f(a) - f(b) = \int_0^1 f'(tb + (1-t)a)\, dt(b - a)$$

to get

$$\begin{aligned}
&\text{(a)} \quad g(\Lambda, x, V) = \int_0^1 g_z(\Lambda, x, tV)\, dtV, \\
&\text{(b)} \quad \qquad\qquad = g_z(\lambda_0, x, 0)\, V + \int_0^1 \int_0^1 g_{zz}(\Lambda, x, stV)\, dst\, dtV^2 \\
&\qquad\qquad\qquad + \int_0^1 \int_0^1 g_{z\lambda}(\lambda_0 + t\epsilon\eta, x, sV)\, ds\, dtV\epsilon
\end{aligned} \tag{3.1}$$

From (3.1a) we obtain, recalling (2.1) (2.10c) and (2.13):

$$\begin{aligned}
|g(\Lambda, x, V)| &\leqslant \| g_z \| \cdot |\epsilon| \cdot (|\phi_0(x)| + |\epsilon| M_0), \\
&\leqslant |\epsilon| (1 + \|\phi_0\|) \| g_z \|.
\end{aligned} \tag{3.1c}$$

Similarly (3.1b) yields

$$\begin{aligned}
&| g(\Lambda, x, V) - g_z(\lambda_0, x, 0)\, V | \\
&\quad \leqslant \epsilon^2 (1 + \|\phi_0\|)[\tfrac{1}{2} \| g_{zz} \| (1 + \|\phi_0\|) + m_0 \| g_{\lambda z} \|].
\end{aligned} \tag{3.1d}$$

Let us denote the numerator in the definition (2.14a) of $\Delta(\Lambda, V)$ by

$$N(\Lambda, V) \equiv \lambda_0 \int_D \phi_0(x)[g(\Lambda, x, V) - g_z(\lambda_0, x, V)]\, dx, \tag{3.2a}$$

and recalling (1.7) the denominator becomes, since v is orthogonal to ϕ_0,

$$\begin{aligned}
D(\Lambda, V) &\equiv \int_D \phi_0(x)\, g(\Lambda, x, V)\, dx, \\
&= \epsilon + \lambda_0^{-1} N(\Lambda, V).
\end{aligned} \tag{3.2b}$$

Using (3.1b) in (3.2a) yields with the definition (2.10a) and the condition (2.12):

$$| N(\Lambda, V) | \leqslant \epsilon^2 \frac{m_0}{2}. \tag{3.3a}$$

With this result and (2.10c) we obtain from (3.2b):

$$\frac{|\epsilon|}{2} \leqslant |D(\Lambda, V)| \leqslant \frac{3|\epsilon|}{2}. \tag{3.3b}$$

These inequalities now yield

$$|\Delta(\Lambda, V)| = \frac{|N(\Lambda, V)|}{|D(\Lambda, V)|} \leqslant |\epsilon| m_0 . \tag{3.3c}$$

From (3.1c, d) and (3.3c) it follows from the definitions (2.14b) and (2.10c) that

$$|R(\Lambda, V)| \leqslant m_0 \left[\frac{1}{2\tilde{D}\|\phi_0\|} + (1 + \|\phi_0\|)\|g_z\|\right]. \tag{3.3d}$$

With $\tilde{\eta}$ and $\tilde{v}(x)$ defined by the transformation T_ϵ in (2.15) we have, from (3.3c),

$$|\tilde{\eta}| \leqslant m_0 ,$$

and from (2.8), (3.3d) and (2.10b)

$$\|\tilde{v}\| \leqslant M_0 .$$

However, $\tilde{v}(x)$ is orthogonal to $\phi_0(x)$ by (2.15d) and from the existence theorem for problems of the form (2.7a, b) and the smoothness conditions (2.11) it follows that $\tilde{v}(x) \in C^{2m+\alpha}(D)$. Thus we have shown that T_ϵ *maps* $\mathscr{I}_0 \times \mathscr{A}_0$ *into itself for each* ϵ *in* $0 < |\epsilon| \leqslant \epsilon_1$.

Now let $\xi \in \mathscr{I}_0$, $w(x) \in \mathscr{A}_0$ and define for each ϵ in $0 < |\epsilon| \leqslant \epsilon_1$:

$$\begin{aligned} W(\epsilon, x) &\equiv \epsilon\phi_0(x) + \epsilon^2 w(x), \\ \Xi(\epsilon) &\equiv \lambda_0 + \epsilon\xi. \end{aligned} \tag{3.4}$$

From the definition (3.2a) we get by expansions similar to that in (3.1b) and upon recalling (1.2b):

$$|N(\Lambda, V) - N(\Xi, W)| \leqslant \epsilon^3 A_1 \|v - w\| + \epsilon^2 A_2 |\eta - \xi| , \tag{3.5a}$$

where

$$\begin{aligned} A_1 &\equiv |\lambda_0|\tilde{D}\|\phi_0\|[(1 + \|\phi_0\|)\|g_{zz}\| + m_0\|g_{\lambda z}\|], \\ A_2 &\equiv |\lambda_0|\tilde{D}\|\phi_0\|(1 + \|\phi_0\|)\|g_{\lambda z}\|. \end{aligned} \tag{3.5b}$$

Similarly, we find that

$$|\lambda_0\{[g_z(\lambda_0, x, 0)\, V - g(\Lambda, x, V)] - [g_z(\lambda_0, x, 0)\, W - g(\Xi, x, W)]\}|$$
$$\leqslant \epsilon^3 \frac{A_1}{\tilde{D}\|\phi_0\|} \|v - w\| + \epsilon^2 \frac{A_2}{\tilde{D}\|\phi_0\|} |\eta - \xi|, \tag{3.5c}$$

$$|g(\Lambda, x, V) - g(\Xi, x, W)|$$
$$\leqslant \epsilon^2[\|g_z\| \cdot \|v - w\| + (1 + \|\phi_0\|)\|g_{\lambda z}\| \cdot |\eta - \xi|]. \tag{3.5d}$$

From (3.2) and (2.14a) we get, using (3.3b):

$$\begin{aligned} |\Delta(\Lambda, V) - \Delta(\Xi, W)| &= \frac{|\epsilon| \cdot |N(\Xi, W) - N(\Lambda, V)|}{|D(\Lambda, V)| \cdot |D(\Xi, W)|}, \\ &\leqslant \frac{4}{|\epsilon|} |N(\Xi, W) - N(\Lambda, V)|. \end{aligned} \tag{3.5e}$$

Combining the inequalities (3.5) with (3.1c) and (3.3c) we obtain from (2.14b)

$$|R(\Lambda, V) - R(\Xi, W)| \leqslant |\epsilon| \frac{B_1}{\tilde{G}} \|v - w\| + \left[|\epsilon| \frac{B_2}{\tilde{G}} + \frac{B_3}{\tilde{G}}\right] |\eta - \xi|, \tag{3.6a}$$

where

$$B_1 \equiv \tilde{G}\left\{\left[\frac{1}{\tilde{D}\|\phi_0\|} + 4(1 + \|\phi_0\|)\|g_z\|\right] A_1 + m_0\|g_z\|\right\},$$
$$B_2 \equiv \tilde{G}(1 + \|\phi_0\|)\, m_0\|g_{\lambda z}\|,$$
$$B_3 \equiv \tilde{G}\left[\frac{1}{\tilde{D}\|\phi_0\|} + 4(1 + \|\phi_0\|)\|g_z\|\right] A_2.$$

Now let

$$T_\epsilon[\xi, w(x)] = [\tilde{\xi}, \tilde{w}(x)].$$

Then from (2.15), (3.5a, e), (2.8) and (3.6a) we find that

$$|\tilde{\eta} - \tilde{\xi}| \leqslant |\epsilon|\, 4A_1 \|v - w\| + 4A_2 |\eta - \xi|,$$

and

$$\|\tilde{v} - \tilde{w}\| \leqslant |\epsilon|\, B_1 \|v - w\| + (|\epsilon|\, B_2 + B_3)|\eta - \xi|.$$

Thus, clearly

$$\max(|\tilde{\eta} - \tilde{\xi}|, \|\tilde{v} - \tilde{w}\|) \leqslant \alpha(\epsilon) \max(|\eta - \xi|, \|v - w\|), \tag{3.7a}$$

where

$$\alpha(\epsilon) \equiv \max\{4[|\epsilon|\, A_1 + A_2], [|\epsilon|(B_1 + B_2) + B_3]\}. \tag{3.7b}$$

Using the condition (2.12) we now find, after some manipulations, that $\alpha(\epsilon) < 1$ if

$$|\epsilon| \leqslant \min\left\{\frac{1}{3m_0}, \frac{1}{4(1 + \|\phi_0\|)M_0}\right\}.$$

However, we have already required that $|\epsilon| \leqslant \epsilon_1$, and so we conclude that $\alpha(\epsilon) < 1$ if $|\epsilon| \leqslant \epsilon_0$.

We have now shown that if $|\epsilon| \leqslant \epsilon_0$ then T_ϵ takes $\mathscr{I}_0 \times \mathscr{A}_0$ into itself and is contracting in the norm: $\max(|\cdot|, \|\cdot\|)$ on $\mathscr{I}_0 \times \mathscr{A}_0$. But this is not quite sufficient to show that the iteration scheme (2.4) and (2.5) converges to a solution of (2.2). Of course it follows from the contraction that the sequence $\{v^{(\nu)}(\epsilon, x)\}$ converges uniformly on $\bar{D}$ and $\{\eta^{(\nu)}(\epsilon)\}$ converges. Also by a simple induction in (2.4)–(2.6) we have that $v^{(\nu)}(\epsilon, x) \in C^{2m+\alpha}(D)$. Thus, we may apply the Compactness Theorem 12.2 of Agmon, Douglis and Nirenberg [1] which justifies taking the limit $\nu \to \infty$ in (2.4)–(2.6). This completes the convergence proof.

It is clear, from the uniform convergence, that the solution

$$\phi_1(\epsilon, x) = \lim_{\nu\to\infty} v^{(\nu)}(\epsilon, x), \qquad \lambda_1(\epsilon) = \lim_{\nu\to\infty} \eta^{(\nu)}(\epsilon) \tag{3.8}$$

satisfies all conditions in (1.8a–c). Also since T_ϵ takes $\mathscr{I}_0 \times \mathscr{A}_0$ into itself it is clear that any initial iterate $[\eta^{(0)}, v^{(0)}(x)] \in \mathscr{I}_0 \times \mathscr{A}_0$ can be used in (2.4) and we obtain the same solution.

We can summarize the above results as the following existence theorem which contains more precise hypothesis than have been indicated thus far:

THEOREM. *Let: L be of order $2m$, uniformly elliptic on $\bar{D}$ with coefficents in $C^\alpha(D)$; B be of order m with coefficients in $C^{m+\alpha}(\partial D)$; the pair (L, B) form a self-adjoint system; the boundary, ∂D, be of class $C^{m+\alpha}$. For all $\lambda \in [\lambda_0 - 1, \lambda_0 + 1]$ and $x \in D$ let $g(\lambda, x, z)$ satisfy* (1.2), (2.11), *and* (2.12) *where: S_0 is defined in* (2.9), *$\epsilon_2 > 0$ is arbitrary, λ_0 is a simple eigenvalue of* (1.6) *with eigenfunction $\phi_0(x)$ normalized as in* (1.7). *Let* (2.8) *hold for all $v(x) \in C^{2m+\alpha}(D)$ and $\psi(x) \in C^\alpha(D)$ satisfying* (2.7a–c). *Then for each ϵ in $0 < |\epsilon| \leqslant \epsilon_0$, with ϵ_0 given in* (2.10), *the problem* (1.1) *has a nontrivial solution of the form* (1.8a–c). *Further $\phi_1(\epsilon, x)$ and $\lambda_1(\epsilon)$ are given in* (3.8) *as the limits of the iterates defined by* (2.4–2.5) *for any initial iterates $[\eta^{(0)}, v^{(0)}(x)] \in \mathscr{I}_0 \times \mathscr{A}_0$ where $\mathscr{I}_0$ and $\mathscr{A}_0$ are defined in* (2.13a, b).

Finally, we point out that estimates of the form (2.8) also follow from the work of Agmon, Douglis and Nirenberg [1]. In fact the estimate

$$\|v\|_{2m+\alpha} \leqslant K \|\psi\|_\alpha$$

follows in close analogy[1] with the proof of Remark 2, p. 669 in [1]. [We need only use a sequence orthogonal to $\phi_0(x)$.] The norms above are not quite those we employ here and we have not attempted to carry out the proof in this generality.

4. Uniqueness

We first show the uniqueness of smooth nontrivial solutions of (1.1) with small norm for λ near λ_0. Then we show how the ambiguity in the choice of sign of the normalized eigenfunction $\phi_0(x)$ is eliminated.

Let $[U(x), \lambda]$ be a nontrivial solution of (1.1) with $U(x) \in C^{2m+\alpha}(D)$ and such that

$$\begin{aligned}
&\text{(a)} && \left| \int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, U(x)\, dx \right| \leqslant \epsilon_0, \\
&\text{(b)} && \| U \| \leqslant (1 + \| \phi_0 \|) \left| \int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, U(x)\, dx \right|, \\
&\text{(c)} && | \lambda - \lambda_0 | \leqslant m_0 \left| \int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, U(x)\, dx \right|.
\end{aligned} \tag{4.1}$$

We will show that $(U(x), \lambda)$ is just the solution represented in (1.8) with the ϵ value

$$\epsilon \equiv \int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, U(x)\, dx. \tag{4.2}$$

It is clear that $\epsilon \neq 0$ or else (4.1b) would imply that $U(x) \equiv 0$, the trivial solution. We can thus define $\psi_1(x)$ and ξ_1 by

$$\begin{aligned}
\psi_1(x) &\equiv \epsilon^{-2}[U(x) - \epsilon\phi_0(x)], \\
\xi_1 &\equiv \epsilon^{-1}[\lambda - \lambda_0].
\end{aligned} \tag{4.3}$$

Since $(U(x), \lambda)$ is a solution of (1.1) it follows that $v \equiv \psi_1(x)$ and $\eta \equiv \xi_1$ must satisfy (2.2) and (2.3) with ϵ given by (4.2). But as T_ϵ is contracting on $\mathscr{I}_0 \times \mathscr{A}_0$ for $|\epsilon| \leqslant \epsilon_0$ it follows that the problem (2.2) and (2.3) has a unique solution with $\eta \in \mathscr{I}_0$ and $v \in \mathscr{A}_0$, for fixed $\epsilon \neq 0$. Thus we need only show that $|\epsilon| \leqslant \epsilon_0$, $\xi_1 \in \mathscr{I}_0$ and $\psi_1(x) \in \mathscr{A}_0$.

From (4.1a) and (4.2) we have $|\epsilon| \leqslant \epsilon_0$. Similarly (4.1c) and (4.2) imply $|\xi_1| \leqslant m_0$ so that $\xi_1 \in \mathscr{I}_0$. The definitions (4.2) and (4.3) yield, upon recalling (1.7),

$$\int_D g_z(\lambda_0, x, 0)\, \phi_0(x)\, \psi_1(x)\, dx = 0.$$

[1] The details of the proof were supplied by R. Lau. By $\| \cdot \|_{r+\alpha}$ we denote the usual norm on $C^{r+\alpha}(D)$, see [1].

Also $\psi_1(x) \in C^{2m+\alpha}(D)$ and we need only verify $\| \psi_1 \| \leqslant M_0$ to conclude that $\psi_1(x) \in \mathscr{A}_0$. However since $\psi_1(x)$ is a solution of (2.2) and satisfies the orthogonality condition (2.3) we have from (2.8):

$$\| \psi_1 \| \leqslant \tilde{G}\epsilon^{-2} \| \lambda_0 [g_z(\lambda_0, x, 0)\, U(x) - g(\lambda, x, U)] - \epsilon \xi_1 g(\lambda, x, U) \| . \tag{4.4}$$

From (4.1), (4.2), and (2.10) we note that $[\lambda, x, U(x)] \in S_0$ for all $x \in \bar{D}$. Then using (1.2a, b) we obtain the bounds

$$\begin{aligned}
&| g(\lambda, x, U)| \leqslant | \epsilon |(1 + \| \phi_0 \|) \| g_z \| ,\\
&|[g(\lambda_0, x, 0)\, U - g(\lambda, x, U)]|\\
&\qquad \leqslant \tfrac{1}{2} \| g_{zz} \| \cdot \| U \|^2 + \| g_\lambda \| \cdot | \lambda - \lambda_0 |\\
&\qquad \leqslant \epsilon^2 (1 + \| \phi_0 \|)[\tfrac{1}{2} \| g_{zz} \| (1 + \| \phi_0 \|) + m_0 \| g_{\lambda z} \|].
\end{aligned}$$

These bounds in (4.4) yield with (2.10) and (2.12) that $\| \psi_1 \| \leqslant M_0$ and the uniqueness proof is concluded.

Suppose we had chosen the normalized eigenfunction $\psi_0(x) \equiv -\phi_0(x)$, in place of $\phi_0(x)$, for our perturbation procedure in section 2. Then using the small parameter $\delta \equiv -\epsilon$ we seek a solution of (1.1) in the form

$$\begin{aligned}
u &= \delta \psi_0(x) + \delta^2 w(x),\\
\lambda &= \lambda_0 + \delta \xi(\delta).
\end{aligned} \tag{4.5}$$

With $[\eta, v(x)]$ a solution of (2.2) and (2.3) it easily follows that (4.5) is a solution of (1.1) if $w(x) \equiv v(x)$ and $\xi(\delta) = -\eta(\epsilon)$. However, if $0 < | \epsilon | \leqslant \epsilon_0$, then $0 < | \delta | \leqslant \epsilon_0$ and by the above uniqueness result there are no other smooth solutions $[\xi, w(x)]$ than that given by $[-\eta, v(x)]$. Thus no additional bifurcated solutions are obtained and the transformation

$$[\epsilon, \phi_0(x)] \to [-\epsilon, -\phi_0(x)]$$

yields the nontrivial solutions already determined.

5. Asymptotic Expansion of $\lambda(\epsilon)$

It is of interest and, in applications, of great importance to study the quantity $\lambda_1(\epsilon)$ of (1.8b) for $| \epsilon |$ near zero. Clearly ϵ is the "amplitude" of thr (linearized) solution near bifurcation and $\lambda(\epsilon)$ might be a load parameter whose deviation from the critical load, λ_0, with amplitude is very significant; see for examples [3]. We shall obtain first order asymptotic representations for $\lambda_1(\epsilon)$ in quite general cases and higher order expansions for important special cases.

From the convergence of the iteration scheme (2.4)–(2.6) it follows that, in the notation (1.8),

$$\begin{aligned} \epsilon\lambda_1(\epsilon) &= \epsilon \lim_{\nu\to\infty} \eta^{(\nu)}, \\ &= \lambda_0 \frac{\epsilon - \int_D \phi_0(x)\, g(\lambda(\epsilon), x, u(\epsilon, x))\, dx}{\int_D \phi_0(x)\, g(\lambda(\epsilon), x, u(\epsilon, x)\, dx} . \end{aligned} \tag{5.1}$$

Employing (1.2a, b) we have, exactly as in the derivation of (3.1),

$$\begin{aligned} g(\lambda, x, u) = g_z(\lambda_0, x, 0)u &+ \int_0^1 \int_0^1 g_{zz}(\lambda_0, x, stu)\, ds\, t\, dt\, u^2 \\ &+ \int_0^1 \int_0^1 g_{\lambda z}(\lambda_0 + t(\lambda - \lambda_0), x, su)\, ds\, dt(\lambda - \lambda_0)u. \end{aligned}$$

Thus, we write

$$\begin{aligned} g(\lambda, x, u) = g_z(\lambda_0, x, 0)\, u + \tfrac{1}{2} g_{zz}(\lambda_0, x, 0)\, u^2 + g_{\lambda z}(\lambda_0, x, 0)(\lambda - \lambda_0)\, u \\ + e_2(x, u)\, u^2 + e_1(\lambda, x, u)(\lambda - \lambda_0)\, u, \end{aligned} \tag{5.2a}$$

where

$$\begin{aligned} e_2(x, u) &\equiv \int_0^1 \int_0^1 [g_{zz}(\lambda_0, x, stu) - g_{zz}(\lambda_0, x, 0)]\, ds\, t\, dt, \\ e_1(\lambda, x, u) &\equiv \int_0^1 \int_0^1 \{g_{\lambda z}[\lambda_0 + t(\lambda - \lambda_0), x, su] - g_{\lambda z}(\lambda_0, x, 0)\}\, ds\, dt. \end{aligned} \tag{5.2b}$$

Since $u(\epsilon, x) = \epsilon\phi_0(x) + \epsilon^2\phi_1(\epsilon, x)$, where ϕ_0 and ϕ_1 satisfy (1.7) and (1.8c), and $\lambda(\epsilon) - \lambda_0 = \epsilon\lambda_1(\epsilon)$ we have from (5.2) that

$$\int_D \phi_0(x)\, g(\lambda(\epsilon), x, u(\epsilon, x))\, dx = \epsilon + \epsilon^2[C_2 + \lambda_1(\epsilon)\, C_1] + \epsilon^3[D_2(\epsilon) + \lambda_1(\epsilon)\, D_1(\epsilon)] \tag{5.3a}$$

where

$$\begin{aligned} C_2 &= \frac{1}{2}\int_D g_{zz}(\lambda_0, x, 0)\, \phi_0^3(x)\, dx, \qquad C_1 \equiv \int_D g_{\lambda z}(\lambda_0, x, 0)\, \phi_0^2(x)\, dx, \\ D_2(\epsilon) &\equiv \int_D \{g_{zz}(\lambda_0, x, 0)\, \phi_1(\epsilon, x) + \epsilon^{-1}\, e_2[x, u(\epsilon, x)][\phi_0(x) + \epsilon\phi_1(\epsilon, x)]\} \\ &\qquad \times [\phi_0(x) + \epsilon\phi_1(\epsilon, x)]\, \phi_0(x)\, dx, \\ D_1(\epsilon) &\equiv \int_D \{g_{\lambda z}(\lambda_0, x, 0)\, \phi_1(\epsilon, x) + \epsilon^{-1}\, e_1[\lambda(\epsilon), x, u(\epsilon, x)][\phi_0(x) + \epsilon\phi_1(\epsilon, x)]^2\} \\ &\qquad \times \phi_0(x)\, dx. \end{aligned} \tag{5.3b}$$

With (5.3a) in (5.1) we obtain, formally,

$$\lambda_1(\epsilon) = \frac{-\lambda_0 C_2 - \epsilon\lambda_0 D_2(\epsilon)}{[1 + \lambda_0 C_1] + \epsilon[C_2 + \lambda_0 D_1(\epsilon) + \lambda_1(\epsilon)\, C_1] + \epsilon^2[D_2(\epsilon) + \lambda_1(\epsilon)\, D_1(\epsilon)]}. \tag{5.4}$$

Although this relation is implicit in $\lambda_1(\epsilon)$ we know that $|\lambda_1(\epsilon)| \leqslant m_0$ for $|\epsilon| \leqslant \epsilon_0$ and this permits the determination of the asymptotic form as $|\epsilon| \to 0$ of $\lambda_1(\epsilon)$.

The expression (5.4) can be written as

$$\lambda_1(\epsilon) = \lambda_1(0) + \epsilon\lambda_2(\epsilon), \tag{5.5a}$$

where we have defined

$$\lambda_1(0) = -\frac{\lambda_0 C_2}{1 + \lambda_0 C_1},$$

$$\lambda_2(\epsilon) \equiv -\frac{\lambda_0 D_2(\epsilon) + \lambda_1(0)[(C_2 + \lambda_0 D_1(\epsilon) + \lambda_1(\epsilon)\, C_1) + \epsilon(D_2(\epsilon) + \lambda_1(\epsilon)\, D_1(\epsilon))]}{(1 + \lambda_0 C_1) + \epsilon(C_2 + \lambda_0 D_1(\epsilon) + \lambda_1(\epsilon)\, C_1) + \epsilon^2(D_2(\epsilon) + \lambda_1(\epsilon)\, D_1(\epsilon))}. \tag{5.5b}$$

If g_{zz} and $g_{\lambda z}$ satisfy Lipschitz conditions of the forms:

$$\begin{aligned} &\text{(a)} \quad |g_{zz}(\lambda_0, x, z) - g_{zz}(\lambda_0, x, 0)| \leqslant K_2 |z|, \qquad \forall(\lambda_0, x, z) \in S_0, \\ &\text{(b)} \quad |g_{\lambda z}(\lambda, x, z) - g_{\lambda z}(\lambda', x, z')| \leqslant K_0 |z - z'| + K_1 |\lambda - \lambda'|, \\ &\qquad\qquad \forall(\lambda, x, z) \;\&\; (\lambda', x, z') \in S_0, \end{aligned} \tag{5.6}$$

then we can show that $\lambda_2(\epsilon) = 0(1)$. More precisely, using (2.10a), (2.12), (5.2b), (5.3b) and (5.6) it can be shown that for $|\epsilon| \leqslant \epsilon_0$:

$$\begin{aligned} &\text{(a)} \quad |C_2| \leqslant \frac{m_0}{4|\lambda_0|}, \qquad |\lambda_0 C_1| \leqslant \tfrac{1}{4}, \qquad |\lambda_1(\epsilon)\, C_1| \leqslant \frac{m_0}{4|\lambda_0|}, \\ &\qquad\qquad |\lambda_1(\epsilon)\, D_1(\epsilon)| \leqslant \frac{m_0}{|\lambda_0|} |\lambda_0 D_1(\epsilon)|, \\ &\text{(b)} \quad |\lambda_0 D_1(\epsilon)| \leqslant m_0 \frac{\|g_{zz}\| + [K_0(1 + \|\phi_0\|) + K_1 m_0](1 + \|\phi_0\|)}{4\|g_{zz}\|(1 + \|\phi_0\|)} \equiv m_0 G_1, \\ &\text{(c)} \quad |D_2(\epsilon)| \leqslant m_0 \frac{6\|g_{zz}\| M_0 + K_2(1 + \|\phi_0\|)^2}{12|\lambda_0| \cdot \|g_{zz}\|(1 + \|\phi_0\|)} \equiv m_0 G_2. \end{aligned} \tag{5.7}$$

We see that $(1 + \lambda_0 C_1) \geqslant \frac{3}{4}$ and so for $|\epsilon|$ sufficiently small the denominator in (5.5b) is, say, at most $\frac{1}{2}$. Specificially let us set

$$\begin{aligned} \epsilon_0' &\equiv \min \left\{\epsilon_0 , \frac{1}{4}\left[\frac{m_0}{|\lambda_0|}\left(\frac{1}{2} + \left[|\lambda_0| + \frac{1}{3}\right] G_1\right) + \frac{1}{3} G_2\right]^{-1}\right\}, \\ m_2 &\equiv 2m_0 \left\{|\lambda_0| G_2 + \frac{1}{3}\left[\frac{m_0}{|\lambda_0|}\left(\frac{1}{2} + \left[|\lambda_0| + \frac{1}{3}\right] G_1\right) + \frac{1}{3} A_2\right]\right\}. \end{aligned} \tag{5.8}$$

Then it follows that $|\lambda_2(\epsilon)| \leqslant m_2$ if $|\epsilon| \leqslant \epsilon_0'$ and (1.9) is proven.

With the first-order asymptotic estimate (5.5a) of $\lambda_1(\epsilon)$, we again consider (5.4) and seek the next order approximation. It is clear that this can be done if the corresponding expansions of $D_2(\epsilon)$ and $D_1(\epsilon)$ are known. However, we note from (5.3b) that this requires, in general, an expansion of $\phi_1(\epsilon, x)$. Indeed, in the perturbation procedure the next order in the expansions of $\lambda(\epsilon)$ and $u(\epsilon, x)$ [equivalently of $\lambda_1(\epsilon)$ and $\phi_1(\epsilon, x)$] are treated simultaneously. However, there are very important (commonly occuring) special cases in which we determine one or more additional orders in the expansion of $\lambda_1(\epsilon)$ with no additional information about $\phi_1(\epsilon, x)$. These cases arise when $g_{zz}(\lambda_0 , x, 0) \equiv 0$ on D and then $\lambda_1(0) = 0$.

Specifically let us assume for some integer $p \geqslant 3$ that

$$g_{z^k}(\lambda_0 , x, 0) \equiv \left.\frac{\partial^k g(\lambda_0 , x, z)}{\partial z^k}\right|_{z=0} \equiv 0 \quad \text{on } D,\ k = 2, 3,..., p - 1; \tag{5.9a}$$

and that the p-th derivative satisfies the weak Lipschitz condition

$$|g_{z^p}(\lambda_0 , x, z) - g_{z^p}(\lambda_0 , x, 0)| \leqslant K_p |z|, \qquad \forall(\lambda_0 , x, z) \in S_0 . \tag{5.9b}$$

A typical example of nonlinearities satisfying these conditions and (1.2) is given by

$$g(\lambda, x, z) \equiv \alpha(\lambda, x)\, z + \beta(\lambda, x,)\, z^p, \qquad p \geqslant 3,$$

where

$$\alpha(\lambda, x) > 0 \qquad \text{on} \qquad \mathscr{I} \times D, \qquad K_p = 0.$$

Using (5.9a) in (5.2b) we can write

$$e_2(x, u)u^2 = g_{z^p}(\lambda_0 , x, 0)\, \frac{u^p}{p!} + e_p(x, u)u^p, \tag{5.10a}$$

where

$$\begin{aligned} e_p(x, u) \equiv \int_0^1 \cdots \int_0^1 [g_{z^p}(\lambda_0 , x, t_1 t_2 \cdots t_p u) - g_{z^p}(\lambda_0 , x, 0)] \\ \times\, t_2 t_3^2 \cdots t_p^{p-1}\, dt_1 \cdots dt_p . \end{aligned} \tag{5.10b}$$

Then, since $p \geqslant 3$, these expressions in (5.3b) with

$$u = u(\epsilon, x) \equiv \epsilon\phi_0(x) + \epsilon^2\phi_1(\epsilon, x)$$

yield

$$D_2(\epsilon) = \epsilon^{p-3}C_p + \epsilon^{p-2}D_p(\epsilon), \tag{5.10c}$$

where

$$C_p \equiv \frac{1}{p!}\int_D g_{z^p}(\lambda_0, x, 0)\, \phi_0^{p+1}(x)\, dx$$

$$D_p(\epsilon) \equiv \int_D \left\{ g_{z^p}(\lambda_0, x, 0) \sum_{\nu=1}^{p} \frac{\epsilon^{\nu-1}\phi_1^{\,\nu}(\epsilon, x)\, \phi_0^{p-\nu-1}(x)}{\nu!\,(p-\nu)!} \right.$$
$$\left. + \epsilon^{-1} e_p[x, u(\epsilon, x)][\phi_0(x) + \epsilon\phi_1(\epsilon, x)]^p \right\} \phi_0(x)\, dx. \tag{5.10d}$$

Recalling that $C_2 = \lambda_1(0) = 0$, since $p \geqslant 3$, we can now write $\lambda_2(\epsilon)$ from (5.5b) as:

$$\lambda_2(\epsilon) = -\frac{\epsilon^{p-3}\lambda_0 C_p + \epsilon^{p-2}\lambda_0 D_p(\epsilon)}{(1 + \lambda_0 C_1) + \epsilon[\lambda_0 D_1(\epsilon) + \lambda_1(\epsilon)\, C_1] + \epsilon^2[D_0(\epsilon) + \lambda_1(\epsilon)\, D_1(\epsilon)]}. \tag{5.11}$$

With the Lipschitz condition (5.9b) in (5.10b) we find that

$$| e_p(x, u)| \leqslant K_p \frac{| u |}{(p+1)!};$$

hence, $| D_p(\epsilon)| = 0(1)$ as $| \epsilon | \to 0$. Thus, exactly as (5.5) was shown to be the asymptotic expansion of $\lambda_1(\epsilon)$ from (5.4), we can now show that the asymptotic expansion of $\lambda_2(\epsilon)$ in (5.11) is given by

$$\lambda_2(\epsilon) = \epsilon^{p-3}\lambda_{p-1}(0) + \epsilon^{p-2}\lambda_p(\epsilon), \tag{5.12a}$$

where

$$\lambda_{p-1}(0) \equiv -\frac{\lambda_0 C_p}{1 + \lambda_0 C_1} \equiv -\lambda_0 \frac{1/p! \int_D g_{z^p}(\lambda_0, x, 0)\, \phi_0^{p+1}(x)\, dx}{1 + \lambda_0 \int_D g_{\lambda z}(\lambda_0, x, 0)\, \phi_0^{2}(x)\, dx}$$

and (5.12b)

$$\lambda_p(\epsilon) \equiv -\frac{\lambda_0 D_p(\epsilon) + \epsilon\lambda_{p-1}(0)\{[\lambda_0 D_1(\epsilon) + \lambda_1(\epsilon)C_1] + \epsilon[D_2(\epsilon) + \lambda_1(\epsilon)D_1(\epsilon)]\}}{(1+\lambda_0 C_1) + \epsilon[\lambda_0 D_1(\epsilon) + \lambda_1(\epsilon)C_1] + \epsilon^2[D_2(\epsilon) + \lambda_1(\epsilon)D_1(\epsilon)]}.$$

Clearly the denominator of $\lambda_p(\epsilon)$ in (5.12b) is not less than $\frac{1}{2}$ for $| \epsilon | \leqslant \epsilon_0'$, given by (5.8). A bound of the form $|\lambda_p(\epsilon)| \leqslant m_p$ is easily obtained. From (1.8b) and (5.5a) we have thus shown that for all $|\epsilon| \leqslant \epsilon_0'$:

$$\lambda(\epsilon) = \lambda_0 + \epsilon^{p-1}\lambda_{p-1}(0) + \epsilon^p\lambda_p(\epsilon), \qquad |\lambda_p(\epsilon)| \leqslant m_p . \tag{5.13}$$

The value of $\lambda_{p-1}(0)$ is easily determined when $\phi_0(x)$ and λ_0 are known. Let us assume that $\lambda_{p-1}(0) > 0$. Then as ϵ traverses the minimum interval of $(-\epsilon_0', \epsilon_0')$ and $\{-[\lambda_{p-1}(0)/m_p], [\lambda_{p-1}(0)/m_p]\}$ the parameter $\lambda(\epsilon)$ traverses an interval in which λ_0 is an endpoint if p is odd and/or λ_0 is an interior point if p is even. Thus for p odd, some small interval above $\lambda = \lambda_0$ [below λ_0 if $\lambda_{p-1}(0)$ is negative] is doubly covered by $\lambda(\epsilon)$. For each value of λ in this doubly covered λ interval the problem (1.1) has two distinct nontrivial solutions of small norm. [These solutions are not $u(\pm\epsilon, x)$ since $\lambda(\epsilon) \neq \lambda(-\epsilon)$ in general. However one of these solutions does correspond to a positive ϵ and the other to a negative ϵ.]

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